

# ON FREELY INDECOMPOSABLE MEASURES

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ABSTRACT. We show that a probability measure is not a nontrivial free convolution if it puts no mass in an interval whose endpoints are atoms. The proof uses analytic subordination.

## 1. INTRODUCTION

Given two probability measures  $\mu, \nu$  on the real line  $\mathbb{R}$ , we denote by  $\mu \boxplus \nu$  their free convolution (see [8] for the definition of free convolution). If  $\nu$  is a point mass, then the measure  $\mu \boxplus \nu$  is just a translation of  $\mu$ . A measure of the form  $\mu \boxplus \nu$ , where neither  $\mu$  nor  $\nu$  is a point mass, is said to be *freely decomposable*. Several classes of measures are known to be freely *indecomposable*. For instance, Belinschi proved in [2, 3] that a measure with nontrivial continuous singular part is freely indecomposable. More recently, Chistyakov and Götze observed in [6] that measures with finite support are freely indecomposable (this result also follows from the description given in [4] of the atoms of a free convolution.) Both of these classes of measures are weakly dense in the set of all Borel probability measures on  $\mathbb{R}$ .

In this note we will prove that  $\mu$  is freely indecomposable if there are points  $\alpha < \beta$  such that  $\mu(\{\alpha\}) > 0$ ,  $\mu(\{\beta\}) > 0$ , and  $\mu((\alpha, \beta)) = 0$ . We also prove analogous results for free multiplicative convolutions  $\boxtimes$  of measures defined on the positive half-line  $\mathbb{R}_+ = [0, +\infty)$ , and on the circle  $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$ .

## 2. ADDITIVE FREE CONVOLUTION

Given a probability measure  $\mu$  on  $\mathbb{R}$ , we define the analytic function  $G_\mu$  on  $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$  by

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t} d\mu(t), \quad \Im z > 0.$$

Note that the measure  $\mu$  is completely determined by the imaginary part of  $G_\mu$ . Set  $\mathbb{C}^- = -\mathbb{C}^+$ . A free convolution  $\mu_1 \boxplus \mu_2$  is characterized analytically by the identity

$$(2.1) \quad G_{\mu_1 \boxplus \mu_2}^{-1}(w) = G_{\mu_1}^{-1}(w) + G_{\mu_2}^{-1}(w) - \frac{1}{w},$$

where  $G_\mu^{-1}$  denote the inverse of  $G_\mu$  relative to composition, and  $w$  belongs to an appropriate Stolz angle at zero in  $\mathbb{C}^-$ , say  $|\Re w| < -\Im w < \varepsilon$  for some  $\varepsilon > 0$ .

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It was shown by Biane [5] (cf. also [9] for an earlier partial result) that, given measures  $\mu_1$  and  $\mu_2$ , there exist analytic functions  $\omega_1, \omega_2 : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  such that

$$(2.2) \quad G_{\mu_1 \boxplus \mu_2}(z) = G_{\mu_1}(\omega_1(z)) = G_{\mu_2}(\omega_2(z)), \quad z \in \mathbb{C}^+.$$

The functions  $\omega_1, \omega_2$  are uniquely determined, and they satisfy

$$\lim_{y \rightarrow +\infty} \frac{\omega_j(iy)}{iy} = 1, \quad j = 1, 2.$$

Moreover, as observed in [4], relation (2.1) can be rewritten as

$$(2.3) \quad \omega_1(z) + \omega_2(z) = z + \frac{1}{G_{\mu_1 \boxplus \mu_2}(z)}, \quad z \in \mathbb{C}^+.$$

The following result is proved in [4].

**Theorem 2.1.** *Assume  $\alpha$  is an atom of the measure  $\mu_1 \boxplus \mu_2$ . Then*

- (1) *the limits  $\alpha_j = \lim_{\varepsilon \downarrow 0} \omega_j(\alpha + i\varepsilon)$  exist,  $j = 1, 2$ .*
- (2)  $\alpha_1 + \alpha_2 = \alpha$ .
- (3)  $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = (\mu_1 \boxplus \mu_2)(\{\alpha\}) + 1$ .
- (4)

$$\lim_{\varepsilon \downarrow 0} \frac{\omega_j(\alpha + i\varepsilon) - \alpha_j}{i\varepsilon} = \frac{\mu_j(\{\alpha_j\})}{(\mu_1 \boxplus \mu_2)(\{\alpha\})}, \quad j = 1, 2.$$

Part (1) actually occurs in the proof of Theorem 7.4 of [4], while (4) is only implicit in that proof. The relevant calculation goes as follows for  $j = 1$ :

$$\frac{\omega_1(\alpha + i\varepsilon) - \alpha_1}{i\varepsilon} = \frac{(\omega_1(\alpha + i\varepsilon) - \alpha_1)G_{\mu_1}(\omega_1(\alpha + i\varepsilon))}{(\alpha + i\varepsilon - \alpha)G_{\mu_1 \boxplus \mu_2}(\alpha + i\varepsilon)}.$$

By Lemma 7.1 in [4], the numerator and denominator of the last fraction converge respectively to  $\mu_1(\{\alpha_1\})$  and  $(\mu_1 \boxplus \mu_2)(\{\alpha\})$  as  $\varepsilon \rightarrow 0^+$ .

**Corollary 2.2.** *Assume that  $\alpha$  and  $\beta$  are atoms of  $\mu_1 \boxplus \mu_2$ , and write them as*

$$\alpha = \alpha_1 + \alpha_2, \quad \beta = \beta_1 + \beta_2$$

*as in the preceding theorem. Then either  $\alpha_1 = \beta_1$  or  $\alpha_2 = \beta_2$ .*

*Proof.* If  $\alpha_1 \neq \beta_1$  and  $\alpha_2 \neq \beta_2$ , then

$$\begin{aligned} 2 &< 2 + (\mu_1 \boxplus \mu_2)(\{\alpha\}) + (\mu_1 \boxplus \mu_2)(\{\beta\}) = \\ &\mu_1(\{\alpha_1\}) + \mu_1(\{\beta_1\}) + \mu_2(\{\alpha_2\}) + \mu_2(\{\beta_2\}) \leq 2, \end{aligned}$$

a contradiction. □

From this point on, we will assume that  $\mu_1$  and  $\mu_2$  are not point masses,  $\mu_1 \boxplus \mu_2$  has two atoms  $\alpha < \beta$ , and  $(\mu_1 \boxplus \mu_2)((\alpha, \beta)) = 0$ . Let us write  $\alpha = \alpha_1 + \alpha_2$  and  $\beta = \beta_1 + \beta_2$  as in Corollary 2.2. Exchanging  $\mu_1$  and  $\mu_2$  if necessary, we may assume that  $\alpha_2 = \beta_2$ .

Furthermore, replacing  $\mu_2$  by  $\mu_2 \boxplus \delta_{-\alpha_2}$ , we may assume that  $\alpha_2 = \beta_2 = 0$  so that  $\alpha = \alpha_1$  and  $\beta = \beta_1$  are atoms of  $\mu_1$  as well as  $\mu_1 \boxplus \mu_2$ .

**Lemma 2.3.** *The functions  $\omega_1$  and  $\omega_2$  can be extended meromorphically across  $(\alpha, \beta)$ . Both continuations are real-valued on  $(\alpha, \beta)$ , with the exception of at most one pole.*

*Proof.* Since  $\omega_1, \omega_2$ , and  $1/G_{\mu_1 \boxplus \mu_2}$  take values in  $\mathbb{C}^+$ , they have Nevanlinna representations

$$\omega_j(z) = r_j + z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma_j(t), \quad j = 1, 2,$$

$$\frac{1}{G_{\mu_1 \boxplus \mu_2}(z)} = s + z + \int_{-\infty}^{\infty} \frac{1+tz}{t-z} d\sigma(t),$$

for  $z \in \mathbb{C}^+$ , where  $r_j, s \in \mathbb{R}$ , and  $\sigma_j, \sigma$  are finite positive Borel measures on  $\mathbb{R}$ . The identity (2.3) implies that  $\sigma_1 + \sigma_2 = \sigma$ . Now, the assumption that  $(\mu_1 \boxplus \mu_2)((\alpha, \beta)) = 0$  implies that  $G_{\mu_1 \boxplus \mu_2}$  can be continued analytically across  $(\alpha, \beta)$ , and this continuation, which we still denote by  $G_{\mu_1 \boxplus \mu_2}$ , is real-valued and strictly decreasing on  $(\alpha, \beta)$ .

Since  $\alpha, \beta$  are atoms of  $\mu_1 \boxplus \mu_2$ , we have

$$\lim_{t \downarrow \alpha} G_{\mu_1 \boxplus \mu_2}(t) = +\infty,$$

and

$$\lim_{t \uparrow \beta} G_{\mu_1 \boxplus \mu_2}(t) = -\infty.$$

Hence, there exists a unique  $\gamma \in (\alpha, \beta)$  so that  $G_{\mu_1 \boxplus \mu_2}(\gamma) = 0$ . It follows that the function  $1/G_{\mu_1 \boxplus \mu_2}$  can be extended meromorphically to  $(\alpha, \beta)$ , with a single simple pole at  $\gamma$ . This means that  $\sigma((\alpha, \gamma)) = \sigma((\gamma, \beta)) = 0$  and  $\sigma(\{\gamma\}) > 0$ . Therefore  $\sigma_j((\alpha, \gamma)) = \sigma_j((\gamma, \beta)) = 0$ , and this implies the claimed properties of  $\omega_j$ .  $\square$

We will need one more detail about the boundary behavior of  $\omega_j$  which is given by the following result.

**Lemma 2.4.** *Let  $\omega : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  be an analytic function and  $\alpha, \gamma \in \mathbb{R}$ . Assume that*

- (1)  *$\omega$  can be continued analytically across  $(\alpha, \gamma)$ ; and also use  $\omega$  to denote this continuation.*
- (2)  *$\omega$  is real-valued on  $(\alpha, \gamma)$ .*
- (3)  *$\lim_{\varepsilon \downarrow 0} \omega(\alpha + i\varepsilon) = \alpha$ .*
- (4) *the limit*

$$a = \lim_{\varepsilon \downarrow 0} \frac{\omega(\alpha + i\varepsilon) - \alpha}{i\varepsilon}$$

*is finite.*

*Then*

$$\lim_{z \rightarrow \alpha, \Re z > \alpha} \frac{\omega(z) - \alpha}{z - \alpha} = a,$$

*In particular,  $\lim_{t \downarrow \alpha} \omega(t) = \alpha$ .*

*Proof.* Note that the limit in (4) always exists, and belongs to  $(0, +\infty]$ . This is called the *Julia-Carathéodory derivative* of  $\omega$  at  $\alpha$  (see Exercises 6 and 7 in [7, Chapter I]). Let us also note that the function  $\omega$  is strictly increasing on  $(\alpha, \gamma)$ . It will be easier to work with the function

$$\tilde{\omega} \equiv \varphi \circ \omega \circ \varphi^{-1},$$

where

$$\varphi(z) = \frac{z - \gamma}{z - \alpha}, \quad z \in \mathbb{C}^+.$$

The assumptions means that

- (1)  $\tilde{\omega}$  can be extended meromorphically to  $(-\infty, 0)$ .
- (2)  $\tilde{\omega}$  is real-valued on  $(-\infty, 0)$ , with the exception of at most one pole, say, at  $t_0 \in (-\infty, 0)$ .
- (3)  $\lim_{y \rightarrow +\infty} \tilde{\omega}(iy) = \infty$ , and
- (4) the limit

$$\lim_{y \rightarrow +\infty} \frac{\tilde{\omega}(iy)}{iy} = \frac{1}{a} \neq 0.$$

The Nevanlinna integral representation for  $\tilde{\omega}$  is therefore

$$\tilde{\omega}(z) = r + \frac{z}{a} + \frac{1 + t_0 z}{t_0 - z} \sigma(\{t_0\}) + \int_0^\infty \frac{1 + tz}{t - z} d\sigma(t), \quad z \in \mathbb{C}^+,$$

where  $r \in \mathbb{R}$ , and  $\sigma$  is a finite positive Borel measure on  $[0, +\infty)$ . Observe now that

$$\left| \frac{t}{t - z} \right| \leq 1, \quad t \in [0, +\infty), \Re z < 0,$$

and the dominated convergence theorem easily yields

$$\lim_{z \rightarrow \infty, \Re z < 0} \frac{\tilde{\omega}(z)}{z} = \frac{1}{a}.$$

This is immediately seen to be equivalent to the conclusion of the lemma.  $\square$

We are now ready for the main result of this section. We denote by  $\delta_t$  the unit point mass at  $t$ .

**Theorem 2.5.** *Let  $\mu_1, \mu_2$  be probability measures on  $\mathbb{R}$ , and  $\alpha < \beta$ . If  $\alpha$  and  $\beta$  are atoms of  $\mu_1 \boxplus \mu_2$ , and  $(\mu_1 \boxplus \mu_2)((\alpha, \beta)) = 0$ , then either  $\mu_1$  or  $\mu_2$  is a point mass.*

*Proof.* Assume to the contrary that neither  $\mu_1$  nor  $\mu_2$  are point masses. We may, and do, assume that  $\alpha$  and  $\beta$  are atoms of the measure  $\mu_1$ . With the notation used earlier, Lemma 2.4 and (2.3) imply that

$$\lim_{t \downarrow \alpha} \omega_2(t) = 0 = \lim_{t \uparrow \beta} \omega_2(t).$$

Since  $\omega_2$  is strictly increasing on  $(\alpha, \gamma)$  and  $(\gamma, \beta)$ , the point  $\gamma$  must really be a pole of  $\omega_2$  so that  $\omega_2((\alpha, \gamma)) = (0, +\infty)$ ,  $\omega_2((\gamma, \beta)) = (-\infty, 0)$ . We will prove that  $\mu_2 = \delta_0$  by showing that  $G_{\mu_2}$  can be continued analytically across  $\mathbb{R} \setminus \{0\}$ , and the

continuation is real-valued on  $\mathbb{R} \setminus \{0\}$ . Indeed, fix a point  $x_0 \in \mathbb{R} \setminus \{0\}$ . There is a unique  $t_0 \in (\alpha, \beta)$ ,  $t_0 \neq \gamma$ , such that  $\omega_2(t_0) = x_0$ . Moreover,  $\omega_2$  is conformal in a neighborhood of  $t_0$ , so that it has an analytic inverse (with respect to composition)  $\varphi$  defined in a neighborhood  $V$  of  $x_0$ , with the property that  $\varphi|_{V \cap \mathbb{R}}$  is real-valued. Therefore, we deduce from (2.2) that

$$G_{\mu_2}(w) = G_{\mu_1 \boxplus \mu_2}(\varphi(w)), \quad w \in V \cap \mathbb{C}^+.$$

Now,  $G_{\mu_1 \boxplus \mu_2}$  is analytic in a neighborhood of  $t_0$ , and therefore the composition  $G_{\mu_1 \boxplus \mu_2} \circ \varphi$  continues analytically to a neighborhood of  $x_0$ . This continuation is real-valued in an interval around  $x_0$  since  $G_{\mu_1 \boxplus \mu_2}$  is real-valued in an interval around  $t_0$ .  $\square$

### 3. MULTIPLICATIVE FREE CONVOLUTION ON $\mathbb{R}_+$

Given a measure  $\mu$  on  $\mathbb{R}_+ = [0, +\infty)$ , different from  $\delta_0$ , we set

$$(3.1) \quad \psi_\mu(z) = \int \frac{tz}{1-tz} d\mu(t),$$

and

$$(3.2) \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

The measure  $\mu$  is determined by the function  $\psi_\mu(z)$  since  $z(\psi_\mu(z) + 1) = G_\mu\left(\frac{1}{z}\right)$ . Note that the function  $\eta_\mu$  is characterized by the properties that  $\eta_\mu(\bar{z}) = \overline{\eta_\mu(z)}$ ,  $\lim_{t \uparrow 0} \eta_\mu(t) = 0$ , and  $\arg \eta_\mu(z) \in [\arg z, \pi)$  for all  $z \in \mathbb{C} \setminus \mathbb{R}_+$ . For two such measures  $\mu_1, \mu_2$ , their free multiplicative convolution  $\mu_1 \boxtimes \mu_2$  is characterized by the relation

$$(3.3) \quad \eta_{\mu_1 \boxtimes \mu_2}^{-1}(w) = \frac{1}{w} \eta_{\mu_1}^{-1}(w) \eta_{\mu_2}^{-1}(w), \quad w < 0.$$

As in the case of additive free convolution, the function  $\eta_{\mu_1 \boxtimes \mu_2}$  is subordinated to  $\eta_{\mu_j}$  (see [5]). More precisely, there exist analytic functions  $\omega_1, \omega_2 : \mathbb{C} \setminus \mathbb{R}_+ \rightarrow \mathbb{C} \setminus \mathbb{R}_+$  such that

$$(3.4) \quad \eta_{\mu_1 \boxtimes \mu_2} = \eta_{\mu_1} \circ \omega_1 = \eta_{\mu_2} \circ \omega_2,$$

and one can also rewrite (3.3) as

$$(3.5) \quad \eta_{\mu_1 \boxtimes \mu_2}(z) = \frac{1}{z} \omega_1(z) \omega_2(z),$$

for all  $z \in \mathbb{C} \setminus \mathbb{R}_+$ . The functions  $\omega_1, \omega_2$  are uniquely determined, and they have the following properties:

- (1)  $\lim_{t \uparrow 0} \omega_j(t) = 0$ ,  $j = 1, 2$ .
- (2)  $\arg z \leq \arg \omega_j(z) < \pi$  for all  $z \in \mathbb{C}^+$ ,  $j = 1, 2$ .
- (3)  $\omega_j(\bar{z}) = \overline{\omega_j(z)}$  for  $z \in \mathbb{C} \setminus \mathbb{R}_+$ ,  $j = 1, 2$ .

The analogue of Theorem 2.1 for free multiplicative convolution is proved in [1].

**Theorem 3.1.** *Let  $\alpha > 0$  be an atom of the measure  $\mu_1 \boxtimes \mu_2$ . Then*

(1) *the limits*

$$\frac{1}{\alpha_j} = \lim_{\varepsilon \downarrow 0} \omega_j \left( \frac{1}{\alpha} + i\varepsilon \right), \quad j = 1, 2,$$

*exist.*

(2)  $\alpha_1 \alpha_2 = \alpha$ .

(3)  $\mu_1(\{\alpha_1\}) + \mu_2(\{\alpha_2\}) = (\mu_1 \boxtimes \mu_2)(\{\alpha\}) + 1$ .

(4)

$$\lim_{\varepsilon \downarrow 0} \frac{\omega_j \left( \frac{1}{\alpha} + i\varepsilon \right) - \frac{1}{\alpha_j}}{i\varepsilon} = \frac{\mu_j(\{\alpha_j\})}{(\mu_1 \boxtimes \mu_2)(\{\alpha\})}, \quad j = 1, 2.$$

Assume now neither  $\mu_1$  nor  $\mu_2$  is a point mass,  $\alpha, \beta \in (0, +\infty)$  are atoms of  $\mu_1 \boxtimes \mu_2$ ,  $\alpha < \beta$ , and  $(\mu_1 \boxtimes \mu_2)((\alpha, \beta)) = 0$ . Then we can write  $\alpha = \alpha_1 \alpha_2$  and  $\beta = \beta_1 \beta_2$  by Theorem 3.1. As in the additive case, we may assume that  $\alpha_2 = \beta_2 = 1$  so that  $\alpha_1 = \alpha$  and  $\beta_1 = \beta$  are atoms of the measure  $\mu_1$ .

**Theorem 3.2.** *Let  $\alpha < \beta$  be two positive real numbers such that  $\alpha$  and  $\beta$  are both atoms for the measure  $\mu_1 \boxtimes \mu_2$ . If  $(\mu_1 \boxtimes \mu_2)((\alpha, \beta)) = 0$ , then either  $\mu_1$  or  $\mu_2$  is a point mass.*

*Proof.* With the above notations, we assume that  $\alpha_1 = \alpha$ ,  $\beta_1 = \beta$ , and  $\alpha_2 = \beta_2 = 1$ . The proof proceeds as that of Theorem 2.5. Thus, assuming that  $\mu_1$  and  $\mu_2$  are not point masses, we show

- (1)  $\eta_{\mu_1 \boxtimes \mu_2}$  continues meromorphically across  $\left( \frac{1}{\beta}, \frac{1}{\alpha} \right)$ ,
- (2)  $\eta_{\mu_1 \boxtimes \mu_2}$  is real-valued on  $\left( \frac{1}{\beta}, \frac{1}{\alpha} \right)$ , with the exception of a simple pole  $\gamma \in \left( \frac{1}{\beta}, \frac{1}{\alpha} \right)$ ,
- (3)  $\omega_1$  and  $\omega_2$  also have continuation properties in (1) and (2),
- (4)  $\omega_2 \left( \left( \frac{1}{\beta}, \gamma \right) \right) = (1, +\infty)$ ,  $\omega_2 \left( \left( \gamma, \frac{1}{\alpha} \right) \right) = (-\infty, 1)$ ,
- (5)  $\eta_{\mu_2}$  is real and analytic on  $\mathbb{R} \setminus \{1\}$ , hence  $\mu_2 = \delta_1$ .

The formula defining  $\psi_{\mu_1 \boxtimes \mu_2}(z)$  makes sense for  $z \in \left( \frac{1}{\beta}, \frac{1}{\alpha} \right)$ , so that (1) and (2) follow immediately from the assumptions on the measure  $\mu_1 \boxtimes \mu_2$ . The proof of (3) is analogous to that of Lemma 2.3. More precisely, we can use the Nevanlinna representation for functions entering the identity

$$\log \omega_1(z) + \log \omega_2(z) = \log \eta_{\mu_1 \boxtimes \mu_2}(z) + \log z, \quad z \in \mathbb{C}^+,$$

where the principal value of the logarithm is used. Property (4) then follows easily from Lemma 2.4, and the fact that  $\omega_2$  must be an increasing function on the intervals  $\left( \frac{1}{\beta}, \gamma \right)$  and  $\left( \gamma, \frac{1}{\alpha} \right)$ . Finally, (5) follows from the relation

$$\eta_{\mu_1 \boxtimes \mu_2}(z) = \eta_{\mu_2}(\omega_2(z))$$

by locally inverting  $\omega_2$  around any point in  $\mathbb{R} \setminus \{1\}$ . □

It should be emphasized that the above result does not hold when  $\alpha = 0$ . An example is provided by the measures

$$\mu_1 = \frac{1}{3}\delta_0 + \frac{2}{3}\delta_1, \quad \mu_2 = \frac{2}{3}\delta_1 + \frac{1}{3}\delta_2.$$

The free convolution  $\mu = \mu_1 \boxtimes \mu_2$  satisfies  $\mu(\{0\}) = \mu(\{1\}) = 1/3$ , while  $\mu((0, 1)) = 0$ . The easiest way to see this is to view  $\mu$  as the distribution of the operator  $p_1(1+p_2)p_1$ , where  $p_1$  and  $p_2$  are freely independent selfadjoint projections in a  $W^*$ -probability space  $(\mathcal{A}, \tau)$ , and  $\tau(p_1) = \tau(p_2) = 1/3$  (We refer to [8] for the notions of a  $W^*$ -probability space and of free independence.)

#### 4. FREE MULTIPLICATIVE CONVOLUTION ON $\mathbb{T}$

For a probability measure  $\mu$  on the unit circle  $\mathbb{T}$ , the functions  $\psi_\mu$  and  $\eta_\mu$  are defined again by (3.1) and (3.2), but their domain of definition is now the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . Assume that

$$\int_{\mathbb{T}} \zeta d\mu_1(\zeta) \neq 0 \neq \int_{\mathbb{T}} \zeta d\mu_2(\zeta).$$

Then the free multiplicative convolution  $\mu_1 \boxtimes \mu_2$  is also characterized by (3.3) in a neighborhood of  $w = 0$ . The subordination functions  $\omega_1, \omega_2$  map  $\mathbb{D}$  to  $\mathbb{D}$ ,  $\omega_1(0) = \omega_2(0) = 0$ , and relations (3.4) and (3.5) are satisfied in  $\mathbb{D}$ . Relation (3.5) is satisfied even when  $\mu_1$  or  $\mu_2$  has first moment equal to zero.

It is proved in [1] that Theorem 3.1 remains valid in this context. The only changes needed in the statement are that the limits must be replaced by radial limits. Thus, the formula in part (1) of Theorem 3.1 becomes

$$\overline{\alpha_j} = \frac{1}{\alpha_j} = \lim_{r \uparrow 1} \omega_j(r\overline{\alpha}),$$

while part (4) becomes

$$\lim_{r \uparrow 1} \frac{\overline{\alpha_j} - \omega_j(r\overline{\alpha})}{(1-r)\overline{\alpha}} = \frac{\mu_j(\{\alpha_j\})}{(\mu_1 \boxtimes \mu_2)(\{\alpha\})}.$$

**Theorem 4.1.** *Let  $\mu_1$  and  $\mu_2$  be two probability measures on  $\mathbb{T}$ , and  $I \subset \mathbb{T}$  be an open arc with endpoints  $\alpha, \beta$ . If  $\alpha$  and  $\beta$  are atoms of  $\mu_1 \boxtimes \mu_2$ , and  $(\mu_1 \boxtimes \mu_2)(I) = 0$ , then either  $\mu_1$  or  $\mu_2$  is a point mass.*

*Proof.* Write  $\alpha = \alpha_1\alpha_2$  and  $\beta = \beta_1\beta_2$ , where  $\alpha_1, \beta_1$  are atoms of  $\mu_1$  and  $\alpha_2, \beta_2$  are atoms of  $\mu_2$ . We may assume that  $\alpha_2 = \beta_2 = 1$  so that  $\alpha_1 = \alpha$  and  $\beta_1 = \beta$ . As in the earlier results, we show that

- (1)  $\eta_{\mu_1 \boxtimes \mu_2}$  continues analytically across  $\overline{I} = \{\zeta : \zeta \in I\}$ ,
- (2)  $|\eta_{\mu_1 \boxtimes \mu_2}(\zeta)| = 1$  for all  $\zeta \in \overline{I}$ ,
- (3)  $\omega_1$  and  $\omega_2$  also have the continuation properties stated in (1) and (2),
- (4)  $\omega_2(\overline{I}) = \mathbb{T} \setminus \{1\}$ ,

- (5)  $\eta_{\mu_2}$  continues analytically across  $\mathbb{T} \setminus \{1\}$ , and  $|\eta_{\mu_2}(\zeta)| = 1$  for  $\zeta \in \mathbb{T} \setminus \{1\}$ .  
Consequently,  $\mu_2 = \delta_1$ .

The continuation in (1) and (2) is given directly by the formula defining  $\eta_{\mu_1 \boxtimes \mu_2}$ . Observe that the zeros of  $\eta_{\mu_1 \boxtimes \mu_2}$  have no accumulation points in  $\bar{I}$ , and therefore the Blaschke product  $B$  corresponding with these zeros is also analytic across  $\bar{I}$ . Let us write the decompositions (see [7, Chapter II])

$$\eta_{\mu_1 \boxtimes \mu_2}(z) = B(z) \exp \left( \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} d\sigma(\zeta) \right),$$

$$\omega_j(z) = B_j(z) \exp \left( \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} d\sigma_j(\zeta) \right), \quad j = 1, 2, \quad z \in \mathbb{D},$$

where  $\sigma$ ,  $\sigma_1$ , and  $\sigma_2$  are finite positive Borel measures on  $\mathbb{T}$ . Relation (3.5) implies that

$$zB(z) = B_1(z)B_2(z),$$

for all  $z \in \mathbb{D}$ , and that

$$\sigma = \sigma_1 + \sigma_2.$$

Thus, the Blaschke products  $B_1$  and  $B_2$  are also analytic across  $\bar{I}$ . Moreover, the fact that  $|\eta_{\mu_1 \boxtimes \mu_2}(\zeta)| = 1$  for  $\zeta \in \bar{I}$  implies that  $\sigma(\bar{I}) = 0$ . We deduce that  $\sigma_1(\bar{I}) = \sigma_2(\bar{I}) = 0$ , and this implies property (3) above. Note that property (3) implies that  $|\omega'_2(\zeta)| \geq 1$  for all  $\zeta \in \bar{I}$ .

The proof of (4) follows from the fact that  $\lim_{\zeta \in \bar{I}, \zeta \rightarrow \bar{\alpha}} \omega_2(\zeta) = \lim_{\zeta \in \bar{I}, \zeta \rightarrow \bar{\beta}} \omega_2(\zeta) = 1$ . To see this, one must use the analogue of Lemma 2.4, which can be proved by using the conformal equivalence between  $\mathbb{C}^+$  and  $\mathbb{D}$ . Finally, (5) follows from the identity

$$\eta_{\mu_1 \boxtimes \mu_2}(z) = \eta_{\mu_2}(\omega_2(z))$$

by locally inverting  $\omega_2$  around any point in  $\mathbb{T} \setminus \{1\}$ . □

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